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Automorphisms of Regular Algebras

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Abstract

Manin associated to a quadratic algebra (quantum space) the quantum matrix group of its automorphisms. This Talk aims to demonstrate that Manin's construction can be extended for quantum spaces which are non-quadratic homogeneous algebras. Here given a regular Artin-Schelter algebra of dimension 3 we construct the quantum group of its symmetries, i.e., the Hopf algebra of its automorphisms. For quadratic Artin-Schelter algebras these quantum groups are contained in the the classification of the $GL(3)$ quantum matrix groups due to Ewen and Ogievetsky. For cubic Artin-Schelter algebras we obtain new quantum groups which are automorphisms of cubic quantum spaces.

All vector spaces and algebras are over a ground field $\mathbb{K}$ of characteristics 0. We adopt the Einstein convention of summing on repeated an upper and a lower indices except when these are in brackets, e.g. there is no summation in $Q^{(i)}_{(i)}$.

1 N-Homogeneous Algebras

A N-homogeneous algebra is an algebra of the form [2], [3]

$$\mathcal{A} = A(E,R) = T(E)/(R)$$

where E is finite dimensional vector space, $T(E)$ is the tensor algebra of E and (R) is the two-sided ideal generated by a vector subspace $R \subset E^{\otimes N}$. Since the space R is homogeneous by ascribing the degree 1 to the generators in E one obtains that the algebra $\mathcal{A}$ is graded, $\mathcal{A} = \oplus_{n\geq 0}\mathcal{A}_n$, generated in degree 1, $\mathcal{A}_0 = \mathbb{K}$ and such that the degrees $\mathcal{A}_n$ are finite-dimensional vector spaces.

The dual $\mathcal{A}^!$ of $\mathcal{A} = A(E,R)$ is defined to be the N-homogeneous algebra $\mathcal{A}^! = A(E^*, R^\perp)$ where E^* is the dual vector space of E and $R^\perp \subset E^{*\otimes N} = (E^{\otimes N})^*$ is the annhilator of R, $R^\perp(R) = 0$. One has $(\mathcal{A}^!)^! = \mathcal{A}$.

Given two N-homogeneous algebras $\mathcal{A} = A(E,R)$ and $\mathcal{A}' = A(E',R')$ one defines the N-homogeneous algebra

$$\mathcal{A} \bullet \mathcal{A}' = A(E \otimes E', \pi_N(R \otimes R'))$$

where π_N is the permutation

$$\pi_N(e_1 \otimes \ldots \otimes e_N \otimes e_1^{'} \otimes \ldots \otimes e_N^{'}) = e_1 \otimes e_1^{'} \ldots e_N \otimes e_N^{'}.$$

Following Manin's monograph [9] on quadratic algebras R. Berger, M. Dubois-Violette and M. Wambst [3] introduced the corresponding semigroup $end(\mathcal{A}) = \mathcal{A}^! \bullet \mathcal{A}$ of the endomorphisms of the N-homogeneous algebra $\mathcal{A}$. The semigroup $end(\mathcal{A})$ is canonically endowed with the structure of bialgebra with a coproduct Δ and counit ε given by

$$\Delta(u^i_j) = u^i_k \otimes u^k_j \qquad \varepsilon(u^j_i) = \delta^j_i. \tag{1}$$

The algebra $\mathcal{A}$ is a left comodule of $end(\mathcal{A}) = \mathcal{A}^! \bullet \mathcal{A}$ for the coaction [3]

$$\delta(x^i) = u^i_j \otimes x^j \qquad u^i_j \in end(\mathcal{A}) = A(E^* \otimes E, r) \qquad r = \pi_N(R^\perp \otimes R). \tag{2}$$

The semigroup $end(\mathcal{A})$ alone has not enough relations in order to allow for an antipode. In order to obtain the quantum matrix group(the group of the automorphisms of $\mathcal{A}$) we shall proceed by "adding the missing relations" [9], considering also the semigroup $end(\mathcal{A}^!)$ of the endomorphisms of the dual $\mathcal{A}^!$. The algebra $\mathcal{A}^!$ becomes a left comodule of the semigroup $end(\mathcal{A}^!) = \mathcal{A} \bullet \mathcal{A}^!$ in the following way. Let us identify the upper and the lower indices in $\mathcal{A}^!$ with the help of the bilinear form $g^{ij} = \delta^{ij}$, $\xi^i = g^{ij}\xi_j = \xi_i$. A left coaction on $\mathcal{A}^!$ is given by

$$\delta(\xi^i) = \check{u}^i_j \otimes \xi^j \quad \check{u}^i_j \in end(\mathcal{A}^!)_g = A(E^* \otimes E, \check{r}) \qquad \check{r} = \pi_N(R \otimes R^\perp)^* \tag{3}$$

where $end(\mathcal{A}^!)_g$ is $end(\mathcal{A}^!)$ up to a move of the indices (by g). We shall not distinguish $end(\mathcal{A}^!)_g$ and $end(\mathcal{A}^!)$, thus $\mathcal{A}^!$ is a left $end(\mathcal{A}^!)$-comodule.

Let us consider the bialgebra $e(\mathcal{A})$(again in the spirit of [9]) having relations those of the semigroups $end(\mathcal{A})$ and $end(\mathcal{A}^!)$, in which the generators u^i_j, $\check{u}^i_j$ have been identified $u^i_j \equiv \check{u}^i_j$, i.e., we consider the bialgebra

$$e(\mathcal{A}) = A(E^* \otimes E, r \oplus \check{r})/(u^i_j - \check{u}^i_j) \qquad \Delta(u^i_j) = u^i_k \otimes u^k_j \qquad \varepsilon(u^j_i) = \delta^j_i \tag{4}$$

which is quotient of the bialgebra $end(\mathcal{A})$ (and $end(\mathcal{A}^!)$). The algebras $\mathcal{A}$ and $\mathcal{A}^!$ are left $e(\mathcal{A})$-comodules in a natural way.

2 Regular Artin-Schelter Algebras

The question that we address in the Talk is when an algebra $\mathcal{A}$ is "good" quantum space, in the sence that the space of its endomorphism $e(\mathcal{A})$ is a Hopf algebra, i.e., there is a quantum group of the symmetries of the quantum space $\mathcal{A}$?

Artin and Schelter have considered a class of regular algebras with very "good" homological properties [1](we use here the equivalent definition of [10]).

Definition 1 *A graded algebra $A = \bigoplus_{n\geq 0} A_n$ with $A_0 = \mathbb{K}$, generated by A_1, $\dim A_1 < \infty$ is called regular if:*

i) A has polynomial growth, (i.e., gk-$\dim A = \gamma < \infty$),

ii) it is Gorenstein, i.e., there is a finite free resolution of the trivial right A-module $\mathbb{K}$, such that its dualized complex (by $Hom_A(\bullet, A)$) is a finite free resolution of the trivial left A-module $\mathbb{K}$. The length of this resolution is called dimension of A.

Manin suggested [10] that the regular algebras are good candidates for quantum spaces. The regular algebras of dimension 2 are exhausted by the Manin plane $yx - qxy = 0$ and the Jordanian plane $xy - yx - y^2 = 0$ (given in the Introducion of [1] as simple examples).

The classification of the regular algebras A of dimension 3 which is done in [1] is much more involved and requires some new technics; it turns out that a regular algebra A of dimension 3 is either quadratic ($N = 2$) or cubic ($N = 3$) homogeneous algebra, $\mathcal{A} = A(E, R)$, generated by two elements satisfying two cubic relations, or else by three elements with three quadratic relations. Further the classification is based on the fact that a regular algebra $\mathcal{A}$ has an intrinsic description in terms of an invariant element $\omega(\mathcal{A})$ of degree $N+1$ in the generators, i.e., in terms of a tensor with $N+1$ indices(Proposition (2.4) of [1]). The invariant tensor $\omega(\mathcal{A})$ transforms as the coordinates of 1-dimensional space of the the maximal non-vanishing degree $\mathcal{A}^!_m$ of $\mathcal{A}^!$

$$\mathcal{A}^!_m = E^* \otimes R^* \cap R^* \otimes E^*, \qquad m = N + 1. \tag{5}$$

Thus the invariant ω of the algebra $\mathcal{A}$ can be written in either of the ways

$$\omega = \xi^i f^*_i = g^*_i \xi^i = Q^j_i f^*_j \xi^i \tag{6}$$

where f_j and $g_i = Q^j_i f_j$ are two bases of R. By change of the basis the matrix Q is amenable to the Jordanian canonical form. Artin and Schelter have proven that the case of Q diagonal matrix is generic, in the sense that the non-diagonal matrices Q do not give new regular algebras. On the components of $\omega(\mathcal{A}) = \omega_I \xi^I \in \mathcal{A}^!_{N+1}$ is defined the cyclic action

$$\sigma(\omega_{Ai}) = \omega_{iA} = Q^i_i \omega_{Ai} \qquad \sigma(\omega_{i_1 \dots i_N i_{N+1}}) = \omega_{i_{N+1} i_1 \dots i_N} = \omega_{\sigma^{-1}(I)}, \tag{7}$$

under which ω splits into orbits. If the multiindex $J = j_1 \dots j_n$ belongs to an orbit of the cyclic action of order n, i.e., if $\sigma^n(J) = J$ then $\omega_J = 0$ or else $Q^{j_1}_{j_1} \dots Q^{j_n}_{j_n} = 1$.

An algebra $\mathcal{A}$ in the classication of the regular algebras of dimension 3 is characterised by the data (Q, ω), the diagonal matrix Q and the invariant ω (see Table (3.9) and (3.11) in [1]) where are listed 6 families of cubic algebras (types A,E,H,S_1,S_2,S'_2) and 7 types of quadratic algebras(types A,B,E,H,S_1,S'_1,S_2).

The cubic class S_1 contains the universal enveloping algebra of the Heisenberg algebra [1] with two generators (then coinciding with the Yang-Mills and related algebras [5], [6]) as well as the algebras related to the parastatistics [7].

3 Koszul Complexes for N-Homogeneous Algebras

To every N-homogeneous algebra $\mathcal{A}$ one associates canonically the Koszul complexes $\mathcal{L}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ [3], [4](see also [5], [6]) as follows.

Let us introduce the element c, independent of the choice of basis

$$c = \xi_i \otimes x^i \in \mathcal{A}^! \otimes \mathcal{A}.$$

and the linear mapping d defined by the left multiplication of c,

$$d : \mathcal{A}^!_n \otimes \mathcal{A} \to \mathcal{A}^!_{n+1} \otimes \mathcal{A} \qquad d : \alpha \otimes a \mapsto c(\alpha \otimes a) = \xi_i \alpha \otimes x^i a.$$

In view of the definition of $R^\perp$ the canonical element satisfies $c^N = 0$, which implies $d^N = 0$. In other words the mapping d is a N-differential mapping and the sequence of spaces $(\oplus_{n\geq 0}\mathcal{A}^!_n \otimes \mathcal{A}, d)$ is a N-complex (for details we send the reader to [3], [4]).

The *Koszul cochain complex* $(\mathcal{L}(\mathcal{A}), \mathfrak{d})$ is defined to be the complex [3]

$$(\mathcal{L}(\mathcal{A}), \mathfrak{d}) = (\bigoplus_{i\geq 0} \mathcal{L}^i(\mathcal{A}), \bigoplus_{i\geq 0} \mathfrak{d}^i)$$

with degrees given by

$$\mathcal{L}^{2i}(\mathcal{A}) = \mathcal{A}^!_{Ni} \otimes \mathcal{A} \qquad \mathcal{L}^{2i+1}(\mathcal{A}) = \mathcal{A}^!_{Ni+1} \otimes \mathcal{A} \tag{8}$$

and differential mapping $\mathfrak{d}^j : \mathcal{L}^j(\mathcal{A}) \to \mathcal{L}^{j+1}(\mathcal{A})$,

$$\mathfrak{d}^j = \begin{cases} d & \text{when} \quad j = 2i, \\ d^{N-1} & \text{when} \quad j = 2i+1. \end{cases}$$

The jump in the degrees (when $N > 2$) $\mathcal{L}^i(\mathcal{A}) = \mathcal{A}^!_{n(i)} \otimes \mathcal{A}$, $n(2i) = Ni$ and $n(2i+1) = Ni+1$, is due to the fact that the complex $\mathcal{L}(\mathcal{A})$ is a contraction of an underlying N-complex [3].

The *Koszul chain complex* $(\mathcal{K}(\mathcal{A}), \mathfrak{d}')$ [3] can be defined as the dualized complex of the cochain complex $\mathcal{L}(\mathcal{A})$, $\mathcal{K}(\mathcal{A}) = Hom_{\mathcal{A}}(\mathcal{L}(\mathcal{A}), \mathcal{A})$ with degrees

$$\mathcal{K}(\mathcal{A}) = \bigoplus_{i\geq 0} \mathcal{K}_i(\mathcal{A}) = \bigoplus_{i\geq 0} \mathcal{A} \otimes \mathcal{A}^{!*}_{n(i)} \tag{9}$$

and differential $\mathfrak{d}'$ defined by d' on $\mathcal{K}_{2i}$ and d'^{N-1} on $\mathcal{K}_{2i+1}$ where d' is the mapping

$$d' : \mathcal{A} \otimes \mathcal{A}^{!*}_k \to \mathcal{A} \otimes \mathcal{A}^{!*}_{k-1} \qquad d' : x_0 \otimes (x_1 \otimes x_2 \otimes \ldots x_n) \mapsto x_0 x_1 \otimes (x_2 \otimes \ldots x_n)$$

for which $d'^N = 0$ holds. The dualization (by $Hom_{\mathcal{A}}(\bullet, \mathcal{A})$) of the Koszul chain complex $\mathcal{K}(\mathcal{A})$ gives back the cochain complex, $\mathcal{L}(\mathcal{A}) = Hom_{\mathcal{A}}(\mathcal{K}(\mathcal{A}), \mathcal{A})$.

It was shown in [3] (using contraction of N-complexes) that the notion of Koszul algebra has a meaningful generalization for N-homogeneous algebras; a N-homogeneous algebra $\mathcal{A}$ is said to be *Koszul algebra* when its Koszul chain complex $\mathcal{K}(\mathcal{A})$ is acyclic in positive degrees [3]. The Koszul chain complex $\mathcal{K}(\mathcal{A})$ of a Koszul algebra $\mathcal{A}$ provides a free resolution of the trivial left module $\mathbb{K}$, property which we shall use right now.

4 Homological Quasi-Determinant

We are ready to introduce the analog of the determinant for a regular Koszul N-homogeneous algebra $\mathcal{A}$. Let $\mathcal{A}$ be N-homogeneous Koszul and regular of dimension D. Then the Koszul chain complex $\mathcal{K}(\mathcal{A})$ (9) of $\mathcal{A}$ provides a free resolution of the left $\mathcal{A}$-module $\mathbb{K}$ with length D and the Gorenstein property implies that the complex $\mathcal{L}(\mathcal{A})$ (8) provides a resolution of the trivial right $\mathcal{A}$-module $\mathbb{K}$, i.e.,

$$H(\mathcal{L}(\mathcal{A})) = H^D(\mathcal{L}(\mathcal{A})) = \mathcal{A}^!_{n(D)} = \mathcal{A}^!_{max} \simeq \mathbb{K},$$

thus the cohomology group $\mathcal{A}^!_{max}$ of $\mathcal{L}(\mathcal{A})$ is a 1-dimensional $e(\mathcal{A})$-comodule.

Definition 2 *The bialgebra $e(\mathcal{A})$ of a regular Koszul N-homogeneous algebra $\mathcal{A}$ coacts on $\mathcal{A}^!_{max}$ by the element $\mathcal{D} = \mathcal{D}(\mathcal{A})$*

$$\delta : \mathcal{A}^!_{max} \to e(\mathcal{A}) \otimes \mathcal{A}^!_{max}, \qquad \delta(\omega(\mathcal{A})) = \mathcal{D}(\mathcal{A}) \otimes \omega(\mathcal{A}) \tag{10}$$

which we are referring to as the homological quasi-determinant.

For the quadratic algebras $N = 2$, the homological quasi-determinant $\mathcal{D}$ coincides with the Manin's *homological determinant* [9].

Lemma 1 *Let us denote the one-dimensional comodule $\mathcal{A}^!_{max}$ as $\omega(\mathcal{A}) = \omega_A \xi^A := \omega_{a_1 \ldots a_m} \xi^{a_1} \ldots \xi^{a_m}$. If the coefficient $\kappa = \omega^A \omega_A$ is invertible then the homological quasi-determinant $\mathcal{D}$ of the bialgebra $(e(\mathcal{A}), \Delta, \varepsilon)$ reads*

$$\mathcal{D} = \kappa^{-1} \omega_A u^A_B \omega^B = \kappa^{-1} \omega_{a_1 \ldots a_m} u^{a_1}_{b_1} \ldots u^{a_m}_{b_m} \omega^{b_1 \ldots b_m} \tag{11}$$

The element $\mathcal{D}$ of $e(\mathcal{A})$ is group-like, $\Delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}$ and $\varepsilon(\mathcal{D}) = 1$.

Proof: In components the coaction of the bialgebra $e(\mathcal{A})$ on the 1-dimensional comodule $\omega \in \mathcal{A}^!_m$ (10) reads

$$\omega_A u^A_B = \mathcal{D} \omega_B. \tag{12}$$

Multiplying of both sides by ω^B and summing on the multi-index B yields

$$\omega_A u^A_B \omega^B = \mathcal{D} \omega_B \omega^B = \kappa \mathcal{D} \tag{13}$$

which for invertible κ implies the quasi-determinant formula (11). The coproduct and the counit on a monomial u^A_B is given by $\Delta(u^A_B) = u^A_C \otimes u^C_B$ and $\varepsilon(u^A_B) = \delta^A_B$ hence

$$\begin{array}{rcccccl} \Delta(\mathcal{D}) & = & \kappa^{-1} \omega_A u^A_C \otimes u^C_B \omega^B & = & \kappa^{-1} \mathcal{D} \omega_C \otimes u^C_B \omega^B & = & \mathcal{D} \otimes \mathcal{D}, \\ \varepsilon(\mathcal{D}) & = & \kappa^{-1} \omega_A \varepsilon(u^A_B) \omega^B & = & \kappa^{-1} \omega_A \delta^A_B \omega^B & = & 1. \qquad \square \end{array}$$

For the quasi-determinant $\mathcal{D}$ there exist an expansion on rows and on columnes which gives rise to the Cramer adjoint elements (analogues of sub-determinants).

Definition 3 *The left $\mathcal{S}_L(u^k_j)$ and right $\mathcal{S}_R(u^k_j)$ Cramer adjoint elements are such elements of the bialgebra $(e(\mathcal{A}), \Delta, \varepsilon)$ that the following holds*

$$\mathcal{S}_L(u^i_k) u^k_j = \mathcal{D} \delta^i_j = \mathcal{D} \varepsilon(u^i_j) = u^i_k \mathcal{S}_R(u^k_j). \tag{14}$$

5 Automorphisms of Regular Algebras of Dimension 3

The regular algebras of dimension 2, i.e., the Manin and the Jordanian plane are Koszul algebras. The quadratic and cubic regular algebras of dimension 3 [1] are also Koszul algebras [2], [4] which allows to define their homological quasi-determinants $\mathcal{D}$ coinciding with the usual quantum determinants [9] for $N = 2$, but giving something new for cubic algebras $N = 3$.

From now on $\mathcal{A}$ will stay for a regular (quadratic or cubic) algebra of dimension 3 of the Artin-Schelter clasification, specified by its data (Q, ω) [1]. We put the accent on the cubic Artin-Schelter algebras, but the quadratic ones fit into the same description, which allows for their simultaneous treatment.

In our construction of the antipode on the bialgebra $e(\mathcal{A})$ the Cramer adjoint elements will be instrumental.

Definition 4 *A regular algebra $\mathcal{A}$ of dimension 3 will be referred to as generic regular algebra when the coefficients κ^i ($\tilde{\kappa}_i$) are invertible*

$$\kappa^i = \omega^{A(i)}\omega_{A(i)} \neq 0 \qquad (\tilde{\kappa}_i = \omega^{(i)A}\omega_{(i)A} \neq 0) \quad \textit{no summation on } (i)!$$

The bialgebra $e(\mathcal{A})$ for a generic $\mathcal{A}$ will be referred to as generic bialgebra.

The equation $\kappa^i = 0$ for some i (equivalent to $\tilde{\kappa}_i = 0$ in view of the cyclicity (7)) singles out the non-generic algebras.

One easily checks that the regular cubic algebras $\mathcal{A}$ of type E,H and some points of the types A, S_1 and S_2 are not generic.

Proposition 1 *Let $\mathcal{A}$ be a generic regular algebra of dimension 3. The left(right) Cramer adjoint elements of the bialgebra $e(\mathcal{A})$ are given by*

$$\mathcal{S}_L(u_i^j) = (\kappa^j)^{-1}\omega_{Ai}u_B^A\omega^{Bj} \qquad (\mathcal{S}_R(u_i^j) = (\,\tilde{\kappa}_i)^{-1}\omega_{iA}u_B^A\omega^{jB}\,). \tag{15}$$

Proof: For a generic bialgebra $e(\mathcal{A})$ one has $\omega^{Ai}\omega_{Aj} = \kappa^i\delta^i_j$ with $\kappa^i \neq 0$ hence

$$\mathcal{S}_L(u_k^i)u_j^k = (\kappa^i)^{-1}\omega_{Ak}u_{Bj}^{Ak}\omega^{Bi} = \mathcal{D}\omega_{Bj}\omega^{Bi}(\kappa^i)^{-1} = \mathcal{D}\delta^i_j = \mathcal{D}\varepsilon(u^i_j).$$

The right Cramer adjoint element are handled similarly due to $\omega^{iA}\omega_{jA} = \tilde{\kappa}_j\delta^i_j$.
□

Proposition 2 *Let $\mathcal{A}$ be a generic regular algebra of dimension 3. The left $\mathcal{S}_L(u^i_j)$ and the right $\mathcal{S}_L(u^i_j)$ Cramer adjoint elements in the bialgebra $e(\mathcal{A})$ are proportional*

$$\mathcal{S}_L(u^i_j) = h^{(i)}_{(j)}\mathcal{S}_R(u^i_j) \tag{16}$$

with coefficient a constant $h^{(i)}_{(j)}$ being an element of the multiplicatively antisymmetric matrix

$$h^{(i)}_{(j)} = \left(\frac{\tilde{\kappa}_{(i)}Q^{(i)}_{(i)}}{\tilde{\kappa}_{(j)}Q^{(j)}_{(j)}}\right)^{-1} = \frac{1}{h^{(j)}_{(i)}}, \qquad h^{(i)}_{(i)} = 1, \qquad h^{(i)}_{(j)}h^{(j)}_{(k)} = h^{(i)}_{(k)}. \tag{17}$$

Proof: Taking into account the cyclicity $\sigma(\kappa^i) = \tilde{\kappa}_i = (Q^{(i)}_{(i)})^2\kappa^{(i)}$ one can bring the expression (15) for the $\mathcal{S}_L(u^i_j)$ to the form of $\mathcal{S}_R(u^i_j)$ which gives the result. $\square$

For the generic regular cubic algebra $\mathcal{A}$ which are our main concern here the formula (17) yields

$$h^{(i)}_{(j)} = h^{j-i}$$

with $h = 1$ for $\mathcal{A}$ of cubic types A and S_1, $h = -1$ for S_2 and $h = -2/3$ for S'_2.

Corollary 1 *The quasi-determinant $\mathcal{D} = \mathcal{D}(\mathcal{A})$ of the bialgebra $(e(\mathcal{A}), \Delta, \varepsilon)$ of a generic regular algebra $\mathcal{A}$ is a quasi-central element*

$$\mathcal{D}u^i_j = h^{(i)}_{(j)} u^i_j \mathcal{D}. \tag{18}$$

Proof: By expansion of $\mathcal{D}$ on the RHS and regrouping the terms we get

$$\begin{aligned} h^{(i)}_{(j)} u^i_j \mathcal{D} &= h^{(i)}_{(k)} u^i_k \mathcal{D}\delta^k_j = h^{(i)}_{(k)} u^i_k \mathcal{S}_L(u^k_s) u^s_j = h^{(i)}_{(k)} h^{(k)}_{(s)} u^i_k \mathcal{S}_R(u^k_s) u^s_j \\ &= h^{(i)}_{(s)} u^i_k \mathcal{S}_R(u^k_s) u^s_j = h^{(i)}_{(s)} \mathcal{D}\delta^i_s u^s_j = h^{(i)}_{(i)} \mathcal{D} u^i_j = \mathcal{D} u^i_j. \end{aligned}$$

Note that the cancelation of the index k in $h^{(i)}_{(k)} h^{(k)}_{(s)} = h^{(i)}_{(s)}$ is crucial for resummation of the terms. $\square$

Theorem 1 *Let $\mathcal{A}$ be a generic regular algebra of dimension 3 [1]. Let us denote by $(\mathcal{H}(\mathcal{A}), \Delta, \varepsilon)$ the bialgebra $(e(\mathcal{A}), \Delta, \varepsilon)$ extended by the inverse element $\mathcal{D}^{-1}$, $\mathcal{D}\mathcal{D}^{-1} = \mathcal{D}^{-1}\mathcal{D} = \mathbb{1}_{e(\mathcal{A})}$ and consider the linear antihomomorphism*

$$S : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})^{op} \qquad S : u^j_i \mapsto S(u^j_i) = \mathcal{D}^{-1}\mathcal{S}_L(u^j_i) = \mathcal{S}_R(u^j_i)\mathcal{D}^{-1}.$$

Then $(\mathcal{H}(\mathcal{A}), \Delta, \varepsilon, S)$ is a Hopf algebra with an antipode given by S.

Proof: The bialgeba structure of $e(\mathcal{A})$ is compatible with the antipode, if they satisfy the antipode axiom

$$m \circ (\mathrm{Id} \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \varepsilon \tag{19}$$

where m is the product and η is the unity mapping of the algebra $e(\mathcal{A})$, $\eta : 1 \mapsto \mathbb{1}_{e(\mathcal{A})}$. The existence of the left $\mathcal{S}_L$ and right $\mathcal{S}_R$ Cramer adjoint elements for $e(\mathcal{A})$ of a generic algebra $\mathcal{A}$(Proposition 1) implies that the antipode S constructed by $\mathcal{S}_L$ or $\mathcal{S}_R$ satisfies the axiom (19) which makes the bialgebra $e(\mathcal{A})$ a Hopf algebra.$\square$

The antipode $S(\mathcal{D})$ of the quasi-determinant $\mathcal{D}$ is evaluated by the axiom (19)

$$S(\mathcal{D})\mathcal{D} = \mathcal{D}S(\mathcal{D}) = \eta \circ \varepsilon(\mathcal{D}) = \mathbb{1}_{e(\mathcal{A})}$$

where we used $\Delta(\mathcal{D}) = \Delta(\mathcal{D}) \otimes \Delta(\mathcal{D})$ and $\varepsilon(\mathcal{D}) = 1$ (Lemma 1). Thus $S(\mathcal{D}) = \mathcal{D}^{-1}$.

6 Conclusions

We have constructed the Hopf algebra $\mathcal{H}(\mathcal{A})$ of the automorphisms of a generic regular algebra $\mathcal{A}$ of dimension 3, or in other words the quantum matrix group for the quantum space $\mathcal{A}$.

The quantum matrix groups $\mathcal{H}(\mathcal{A})$ for a quadratic $\mathcal{A}$ are contained in the Ewen and Ogievetsky classification [8] (see also [11]) of the $GL(3)$ quantum matrix group. The quantum groups $\mathcal{H}(\mathcal{A})$ for the cubic regular algebras $\mathcal{A}$ of dimension 3 are to the best of our knowledge new ones (first reported by the author in [12]).

One expects that all $\mathcal{H}(\mathcal{A})$ have polynomial growth. It is natural for the automorphism group $\mathcal{H}(\mathcal{A})$ of an algebra $\mathcal{A}$ with dimension 3 to expect gk-$\dim \mathcal{H}(\mathcal{A}) = 9 = 3^2$. as it happens for the quadratic algebras. Surprisingly one has gk-$\dim \mathcal{H}(\mathcal{A}) = 7$ for some cubic $\mathcal{A}$, e.g. of type S_2.

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